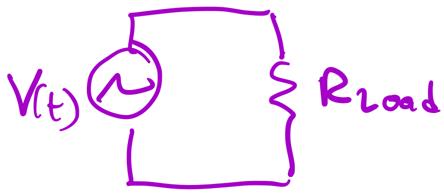
AC Circuits



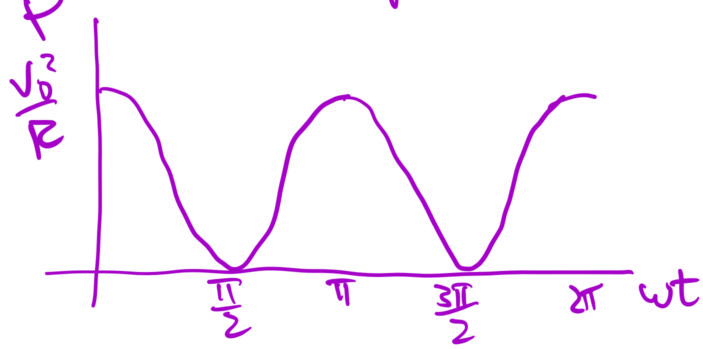
$V(t) = V_0 \cos\omega t \qquad \omega = 2\pi f$

by Ohm's law: $V = IR$

$\therefore \quad I(t) = \frac{V_0}{R}\cos\omega t$

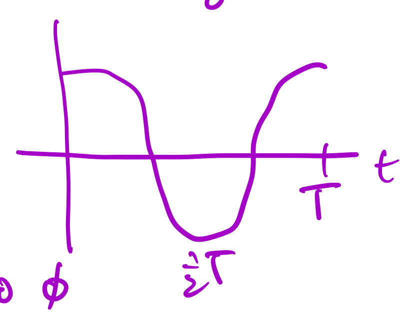
current & voltage are in phase.
why important?

$$\text{Power} = IV = \frac{V_0^2}{R}\cos^2\omega t$$



Power oscillates. Average power: $\bar{P} = \frac{1}{T}\int_0^T P(t)\,dt \qquad \omega T = 2\pi$

$$\bar{P} = \frac{1}{T}\frac{V_0^2}{R}\underbrace{\int_0^T \cos^2\omega t\,dt}_{\frac{1}{2}(1+\underbrace{\cos 2\omega t}_{\text{integrates to } \phi})}$$



so $\bar{P} = \frac{1}{T}\frac{V_0^2}{R}\frac{T}{2}$

$= \frac{1}{2}\frac{V_0^2}{R} \qquad$ ave power $= \frac{1}{2}$ max power

RMS root mean square

$V_{rms} = \sqrt{\overline{V^2}}$ note: $\overline{V^2} \neq \overline{V}^2$

$\overline{V^2} = \frac{1}{T}\int_0^T V^2(t)\,dt = \frac{V_0^2}{T}\int_0^T \cos^2 \omega t\,dt = \frac{V_0^2}{2}$

$\therefore V_{rms} = \frac{V_0}{\sqrt{2}}$ ← max voltage

similarly $I_{rms} = \frac{I_0}{\sqrt{2}} = \frac{V_0}{\sqrt{2}R}$

$\therefore \overline{P} = \frac{1}{2}\frac{V_0^2}{R} = \frac{1}{2}V_0 \cdot \frac{V_0}{R} = \frac{V_0}{\sqrt{2}} \cdot \frac{I_0}{\sqrt{2}} = V_{rms} I_{rms}$

in home $V_{rms} = 120$ Volts

if you draw 10A, that's I_{rms}

so $\overline{P} = 120 \cdot 10 = 1.2\,kW$

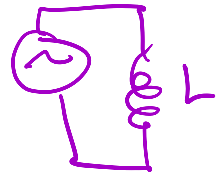
$V = IR + L\frac{dI}{dt}$

$IR - V = L\frac{dI}{dt}$

$I - V/R = \frac{L}{R}\frac{dI}{dt}$

$I - I_0 = \tau\frac{dI}{dt}$ $\tau = L/R$

AC w/ Inductor



$V = V_0 \cos\omega t \rightarrow V_0 e^{i\omega t}$

$V_L = L\frac{dI}{dt} = V$

$\frac{dI}{dt} = \frac{V}{L} = \frac{V_0}{L} e^{i\omega t}$

$I = \frac{V_0}{i\omega L} e^{i\omega t}$

"resistance"

$X_L = i\omega L$ reactance of ind.

↳ complex impedance

note: as $\omega \rightarrow \infty$ inductor impedance $\rightarrow \infty$

so inductor is "low pass filter"

lets low freq signals thru

to get $I(t)$ keep only real part

$I = \frac{V_0}{i\omega L} e^{i\omega t} = -\frac{i V_0}{\omega L}(\cos\omega t + i\sin\omega t)$

$= \frac{V_0}{\omega L}\sin\omega t - \frac{i V_0}{\omega L}\cos\omega t$

throw away $\mathrm{Im}(I)$

$V = V_0 \cos\omega t$

$I = \frac{V_0}{\omega L}\sin\omega t = \frac{V_0}{\omega L}\cos(\omega t - \pi/2)$

current "lags" voltage by 90°

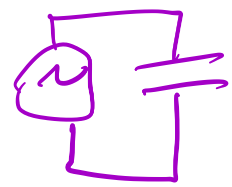
average power: $P_{av} = \frac{1}{T}\int_0^T I(t)V(t)\,dt$

$$= \frac{1}{T}\frac{V_0^2}{\omega L}\int_0^T \sin\omega t \cos\omega t = 0$$

no power loss in (ideal) inductor (assumes $R_L = 0$)

AC w/capacitor

$V = V_0 e^{i\omega t}$



$C = Q/V$

$V_c = \frac{Q}{C} \Rightarrow Q = V_c C = VC$

$$I = \frac{dQ}{dt} = C\frac{d}{dt}V_0 e^{i\omega t} = V_0\, i\omega C e^{i\omega t} = \frac{V_0}{X_c} e^{i\omega t}$$

$X_c = \frac{1}{i\omega C}$ capacitor $X_c \to \infty$ as $\omega \to 0$ DC open

capacitor filters out low freq, passes high "high pass filter"

$$I = V_0 i\omega C e^{i\omega t} = V_0 \omega C\, i(\cos\omega t + i\sin\omega t)$$
$$= -\frac{V_0}{1/\omega C}\sin\omega t$$

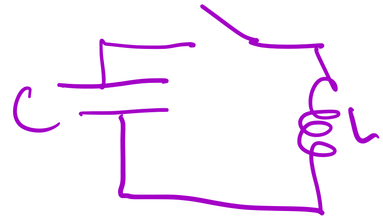
P_c also zero for same reason as capacitor

$$-\sin\omega t = \cos(\omega t + \alpha) = \cos\omega t\cos\alpha - \sin\omega t\sin\alpha$$
$$\alpha = \pi/2$$

so $I = \frac{V_0}{1/\omega C}\cos(\omega t + \pi/2)$

current "leads" voltage by $\pi/2$

LC



charge up plates, Q

voltage will be $V = Q/C$

close switch: $\frac{Q}{C} = -L\frac{dI}{dt}$

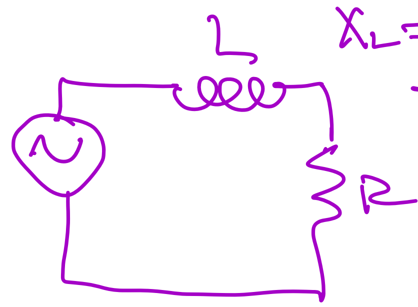
$$\frac{Q}{C} + L\frac{dI}{dt} = 0$$

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0 \qquad Q = Q_0\cos\omega_0 t \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$\omega_0 \Rightarrow$ natural freq of oscillation

no power loss in C or L ∴ energy just oscillates between E^2 in cap & B^2 in inductor

AC and LR



$X_L = \omega L$

$Z_L = i\omega L$
$= iX_L$

$V = V_0 e^{i\omega t} = IZ$

Z: complex impedance

$= R + Z_L = R + i\omega L$

so $I = \frac{V_0 e^{i\omega t}}{R + i\omega L}$

write $R + i\omega L = \sqrt{R^2 + (\omega L)^2}\, e^{i\phi}$ $\qquad \tan\phi = \frac{\omega L}{R}$

$= R\sqrt{1 + \left(\frac{\omega L}{R}\right)^2}$

let $\tau = L/R$ decay time for L in DC circuit

then $I = \dfrac{V_0 e^{i\omega t} e^{-i\phi}}{R\sqrt{1+(\omega\tau)^2}} = \dfrac{V_0 \cos(\omega t - \phi)}{R\sqrt{1+(\omega\tau)^2}}$

can write $I_0(\omega) = \dfrac{V_0}{R} \dfrac{1}{\sqrt{1+(\omega\tau)^2}}$

$I(t) = I_0(\omega)\cos(\omega t - \phi)$ $\quad \tan\phi = \dfrac{\omega L}{R} = \dfrac{X_L}{R}$

ratio complex/real impedance

$$\overline{P}_{\text{delivered}} = \overline{IV} = \frac{V_0^2}{R}\frac{1}{\sqrt{}}\frac{1}{T}\int_0^T \cos(\omega t - \phi)\cos\omega t\, dt$$

$$= \frac{V_0^2}{R}\frac{1}{\sqrt{}}\frac{1}{T}\int \left(\cos\omega t\cos\phi + \cancel{\sin\omega t \sin\phi}\right)\cos\omega t\, dt$$

$$= \frac{V_0^2}{2R}\frac{1}{\sqrt{1+(\omega\tau)^2}}\cos\phi$$

$$\overline{P}_R = \overline{I^2 R} = \frac{1}{T}\frac{V_0^2 R}{R^2(1+(\omega\tau)^2)}\int_0^T \cos^2(\omega t - \phi)\, dt = \frac{V_0^2}{2R(1+(\omega\tau)^2)}$$

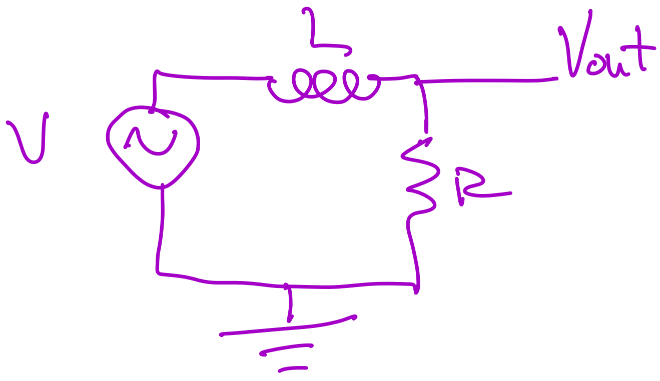
$$\overline{P}_L = \overline{IL\frac{dI}{dt}} = \overline{\frac{V_0\cos(\omega t - \phi)}{R\sqrt{}}\frac{L(-\omega)V_0 \sin(\omega t - \phi)}{R\sqrt{}}} = 0$$

$\overline{P}_{del} = \overline{P}_R$ so $\cos\phi = \dfrac{1}{\sqrt{1+(\omega\tau)^2}}$

$\tan\phi = \dfrac{\omega L}{R} = \omega\tau \Rightarrow 1 + \tan^2\phi = \sec^2\phi = 1 + (\omega\tau)^2$ ✓

$I(\omega) \to 0$ as $\omega \to \infty$ because L is low pass filter

We can ground part of the circuit to set EM potential to be zero there.



$\Rightarrow$ Voltage V_{out} = voltage across resistor

$$V_{out} = IR = \frac{V_0 \cos(\omega t - \phi)}{\sqrt{1 + (\omega \tau)^2}}$$

as $\omega \to 0$, $V_{out} \to V_0$ low pass

$\omega \to \infty$, $V_{out} \to 0$ filters out high frequencies

now swap L & R

voltage across inductor:

$$V_{out} = V_L = I X_L = \frac{i V_0 \omega L\, e^{i(\omega t - \phi)}}{R\sqrt{1+(\omega\tau)^2}}$$

$$Re(V_{out}) = Re\left(\frac{V_0 \omega L i}{R\sqrt{\quad}}\left[\cos(\omega t-\phi) + i\sin(\omega t-\phi)\right]\right)$$

$$V_{out} = \frac{-V_0 \omega L \sin(\omega t-\phi)}{R\sqrt{\quad}}$$

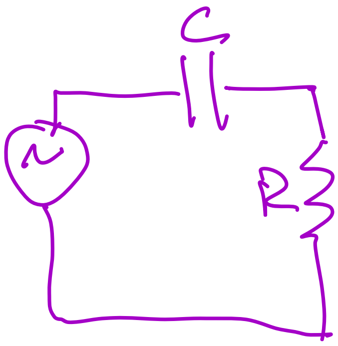
since ω is in numerator, $V_{out} \to 0$ as $\omega \to 0$

as $\omega \to \infty$, $\frac{\omega L}{\sqrt{1+(\omega\tau)^2}} \to R$, $V_{out} \to V_0$

this filters out low frequencies!

$\Rightarrow$ basically, inductor allows low freq to pass to ground leaving high frequencies

AC RC circuits



$X_c = \frac{1}{\omega C}$, $Z_c = \frac{1}{i\omega C}$

$$Z = R + Z_c = R + \frac{1}{i\omega C} = \sqrt{R^2 + \frac{1}{(\omega C)^2}}\, e^{i\phi}$$

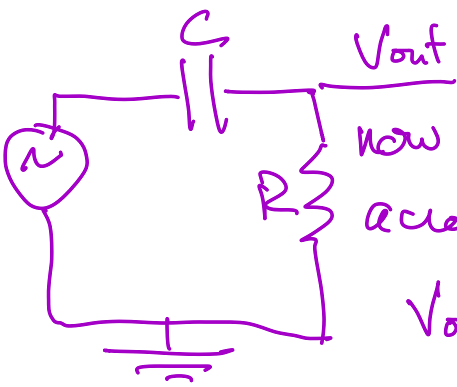
$$\tan\phi = \frac{1}{\omega RC}$$

here RC = decay constant for RC discharge

let $\tau = RC$ and write $\sqrt{R^2 + \frac{1}{(\omega C)^2}} = R\sqrt{1 + \frac{1}{(\omega\tau)^2}}$

$$\text{so } I = \frac{V}{Z} = \frac{V_0 e^{i\omega t}}{R\sqrt{1 + 1/(\omega\tau)^2}\, e^{i\phi}} = \frac{V_0}{R}\frac{e^{i(\omega t - \phi)}}{\sqrt{1 + 1/(\omega\tau)^2}}$$

similar to RL!



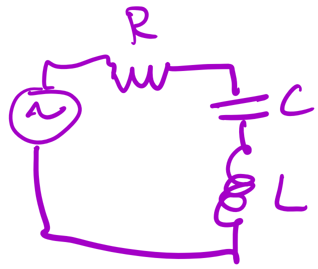
now ground circuit and look at V_{out} across resistor:

$$V_{out} = IR = \frac{V_0 \cos(\omega t - \phi)}{\sqrt{1 + 1/(\omega\tau)^2}}$$

as $\omega \to \infty$ $V_{out} \to V$ high pass filter

swap R & C to get low pass filter

LCR



$V = V_0 \cos\omega t \to V_0 e^{i\omega t}$

complex impedance Z

$$Z = R + i\omega L + \frac{1}{i\omega C}$$

$$= R + i(\omega L - 1/\omega C)$$

write Z in exponential form $Z = r e^{i\phi}$

where $r = \sqrt{R^2 + (\omega L - 1/\omega C)^2}$ and $\tan\phi = \frac{\omega L - 1/\omega C}{R}$

then use "Ohm's" law equivalent $V = IZ$

solve for $I = \frac{V}{Z} = \frac{V_0 e^{i\omega t}}{r e^{i\phi}} = \frac{V_0}{r} e^{i(\omega t - \phi)}$

now keep real part to get

$$I = \frac{V_0}{r}\cos(\omega t - \phi) = \frac{V_0}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}\cos(\omega t - \phi)$$

write $I(\omega) \equiv \frac{V_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$ gives

$$I(t) = I(\omega)\cos(\omega t - \phi)$$

properties of $I(\omega)$:

$I(0) = 0$ ($1/\omega C$ term dominates)

$I(\infty) = 0$ (ωL " ")

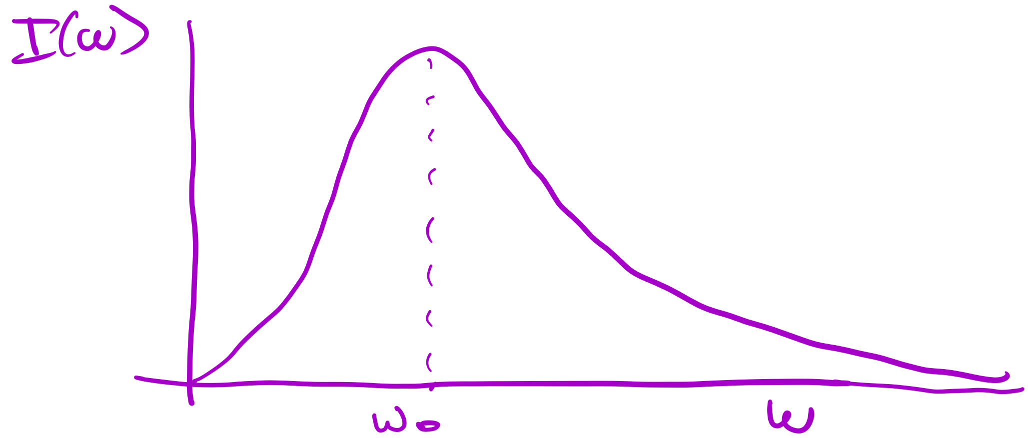
$I(\omega_m)$ is a max when $\omega_m L - \frac{1}{\omega_m C} = 0$

or $\omega_m = 1/\sqrt{LC}$

this is the resonance freq of LC part

$\omega_m = \omega_0 = 1/\sqrt{LC}$

Resonance! when drive system at $\omega = \omega_0$, maximizes response



note as $R \to 0$, $I(\omega_0) \to \infty$ so its more "peaked"

$\Rightarrow$ this happens when you minimize R, or reduce energy loss thru resistance

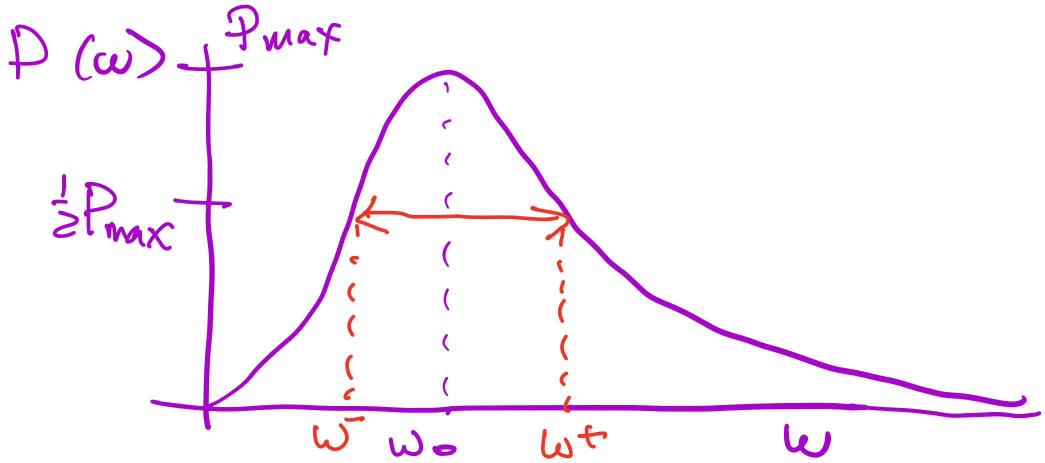
can characterize how wide or peaked signal is

P dissipated is thru resistor (load)

$$P = I^2 R = \frac{V_0^2 \cos^2(\omega t - \theta)}{R^2 + (\quad)^2} \cdot R$$

max is at $P_{max} = \frac{V_0^2}{R}$

find ω where $P = \frac{1}{2} P_{max}$



can calculate ω^- & ω^+

$$P(\omega) = \frac{1}{2} P_{max} = \frac{V_0^2}{2R} = \frac{V_0^2 R}{R^2 + (\omega L - 1/\omega C)^2}$$

so $\quad 2R^2 = R^2 + (\quad)^2$

$$R^2 = (\omega L - 1/\omega C)^2 \quad \text{or} \quad \omega L - \frac{1}{\omega C} = \pm R$$

2 solutions as expected

for $+R$ solution: $\omega L - \frac{1}{\omega C} = R$

$$\omega^2 L - R\omega - 1/C = 0$$

$$\omega^2 - \frac{R}{L}\omega - \frac{1}{LC} = 0$$

write as $\quad \omega^2 - \omega/\tau - \omega_0^2 = 0 \qquad \tau = L/R$

$$\omega_0^2 = 1/LC$$

solves as $\omega = \dfrac{\frac{1}{\tau} \pm \sqrt{\frac{1}{\tau^2} + 4\omega_0^2}}{2}$

only "+" solution gives $\omega > 0$ so write

$$\omega^+ = \frac{1}{2\tau} + \sqrt{\frac{1}{4\tau^2} + \omega_0^2}$$

after more math can show other solution

$$\omega^- = \frac{-1}{2\tau} + \sqrt{\frac{1}{4\tau^2} + \omega_0^2}$$

$$\Delta\omega = \omega^+ - \omega^- = \frac{1}{2\tau} + \sqrt{\quad} - \left(-\frac{1}{2\tau} + \sqrt{\quad}\right)$$

$$\Delta\omega = \frac{1}{\tau} = \frac{R}{L}$$ ⇐ makes sense that $\Delta\omega \sim R$.

quality factor "Q" $= \dfrac{\omega_0}{\Delta\omega}$ how peaked

$Q = \dfrac{\omega_0}{\Delta\omega} = \omega_0\tau = \dfrac{\omega_0 L}{R}$ ⇒ ratio of complex to real impedance

$= \dfrac{X_L}{R}$

want $Q \to \infty$ for highly selective filtering

note at resonance, $\omega_0 L = \dfrac{1}{\omega_0 C}$ so $X_L = X_C$

so $Q = \dfrac{X_C}{R} = \dfrac{X_L}{R}$ either way average power delivered gets dissipated only thru real resistance R

$$\bar{P} = \frac{\int_0^T P(t)\,dt}{T} \qquad \text{where } \omega T = 2\pi$$

let $I(\omega) = \frac{V_0}{\sqrt{\ }}$

$$= \frac{1}{T}\int_0^T V_0 I(\omega)\cos\omega t \cos(\omega t - \phi)\,dt$$

$$= V_0 I(\omega)\int \cos\omega t\,[\cos\omega t\cos\phi + \sin\omega t\sin\phi]\,dt$$

note: $\int_0^T \cos\omega t \sin\omega t\,dt = \frac{1}{2}\int_0^T \sin 2\omega t\,dt$

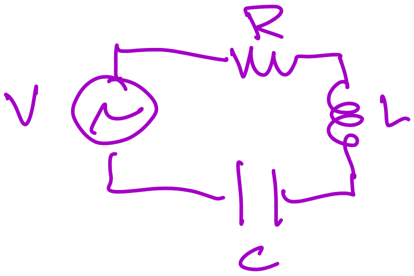
$$= \frac{1}{4\omega}\ -\cos\omega t\,\Big|_0^T = \frac{1}{4\omega}\left[1 - \cos 2\pi\right] = 0$$

$$\int \cos^2\omega t\,dt = \int_0^T \frac{1}{2}(1 + \cos 2\omega t)\,dt = \frac{T}{2}$$

$$\therefore\ \bar{P} = \frac{V_0 I(\omega)\cos\phi}{2} = \underbrace{\frac{V_0}{\sqrt{2}}}_{V_{rms}}\ \underbrace{\frac{I(\omega)}{\sqrt{2}}}_{I_{rms}}\cos\phi$$

$$\bar{P} = V_{rms} I_{rms} \underbrace{\cos\phi}_{\text{"power factor"}}$$

LCR circuits - diff. eqn's



$V = V_0 \cos\omega t = V_R + V_L + V_C$

$V_R = IR = R\frac{dQ}{dt}$

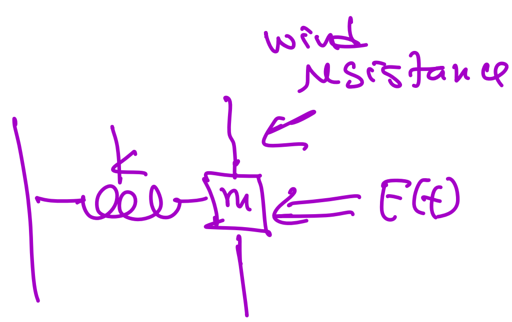
$V_L = L\frac{dI}{dt} = L\frac{d^2Q}{dt^2}$

$V_C = \frac{Q}{C}$

diff. eqn is: $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V_0 \cos\omega t$

solve for $Q(t)$

$\Rightarrow$ this is the same diff eq for a mass on a spring that is driven by a force $F(t)$ w/ wind resistance



$F_{spring} = -kx$

$F_{resist} = -bv = -b\frac{dx}{dt}$

$ma = m\frac{d^2x}{dt^2} = -kx - bv + F(t)$

or $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t)$

same equation!

$m \Leftrightarrow L$ inductance ("inertia")

$b \Leftrightarrow R$ resistance (energy dissipation)

$k \Leftrightarrow \frac{1}{C}$ "elastic" force

$F(t) \Leftrightarrow V(t)$ driver

so we know $x(t) = X(\omega)\cos(\omega t - \phi)$

and $\omega_0^2 = \frac{1}{LC} \Rightarrow \frac{k}{m}$

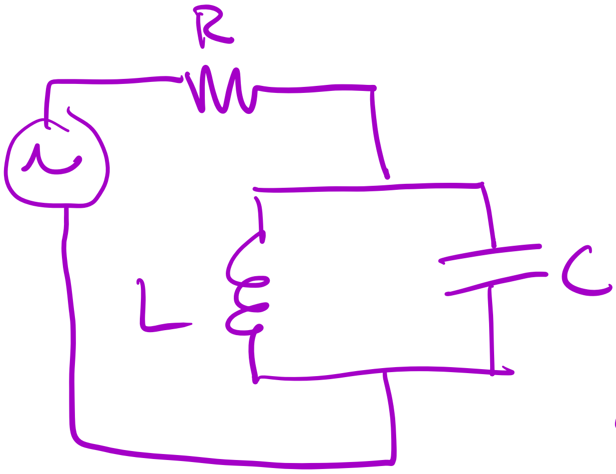
and $x(\omega)$ has same form as above

if you drive w/ $\omega = \omega_0$, amplitude is maximized

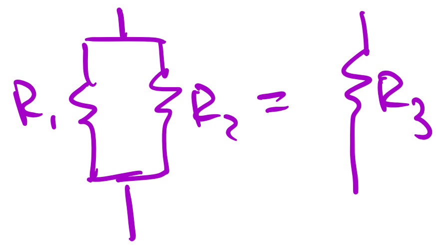
quality factor $Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0 L}{R} \Rightarrow \frac{\omega_0 m}{b}$

as $b \to 0$, $x(\omega)$ becomes narrower

Parallel LCR

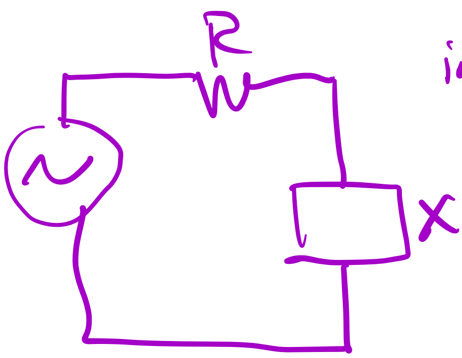


remember with resistors:



and $\frac{1}{R_3} = \frac{1}{R_1} + \frac{1}{R_2}$ parallel

so the complex impedance of the parallel L, C looks the same as for real impedances



impedance of "X" is:

$$\frac{1}{X} = \frac{1}{X_L} + \frac{1}{X_C} = \frac{1}{i\omega L} + i\omega C$$

$$= i\left(\omega C - \frac{1}{\omega L}\right)$$

$$X = \frac{1}{i\left(\omega C - \frac{1}{\omega L}\right)} = \frac{-i}{\left(\omega C - \frac{1}{\omega L}\right)}$$

we analyze this just like the series LCR

$$V = V_0 e^{i\omega t}$$

$$I = \frac{V}{Z}$$

$Z = R - i r$ where $r = \dfrac{1}{\omega C - \frac{1}{\omega L}}$

we write $Z = \sqrt{R^2+r^2}\, e^{i\phi}$

$$\text{so } I(t) = \frac{V_0 e^{i\omega t}}{\sqrt{R^2+r^2}\, e^{i\phi}} = \frac{V_0 e^{i(\omega t-\phi)}}{\sqrt{}}$$

keeping only the real parts:

$$I(t) = \frac{V_0 \cos(\omega t-\phi)}{\sqrt{R^2+r^2}}$$

but $r = \dfrac{1}{\omega C - \frac{1}{\omega L}}$ parallel LCR

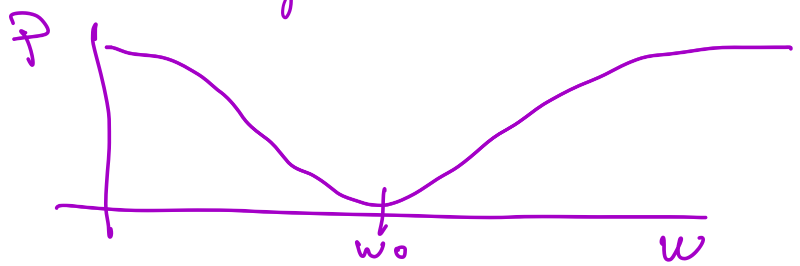
$r = \omega L - \dfrac{1}{\omega C}$ series

$r \to 0$ as $\omega \to \omega_0 = 1/\sqrt{LC}$ so $1/r \to \infty$

note: for series LCR, $\omega = \omega_0$ maximizes $I(\omega)$
for parallel $\omega = \omega_0$ <u>minimizes</u> "

$$\text{Power} = I^2 R = \frac{V_0^2 R}{R^2+r^2}\cos^2(\omega t-\phi)$$

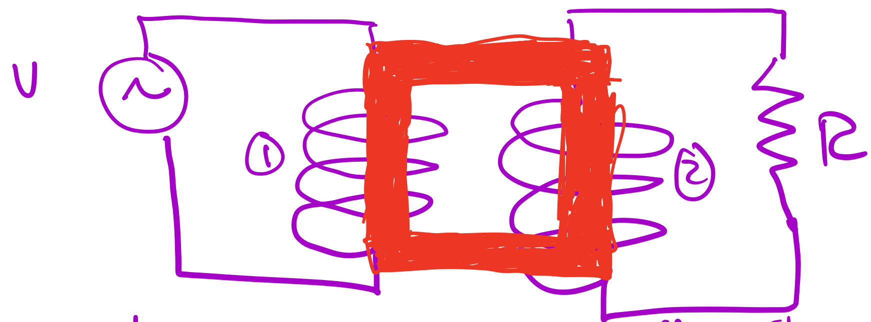
power is minimized at $\omega = \omega_0$!



Transformers

Faraday's law for coils w/ N turns

$V_1 = V_0 \cos \omega t$



put iron core thru both coils, and connect them
B will run around the core

coil 1 ≡ "primary"
coil 2 ≡ "secondary

in primary, $\Phi_1 = B_1 A_1$
in secondary, $\Phi_2 = B_2 A_2$

$B_1 \neq B_2$! $B \propto$ # field lines

but $\Phi_1 = \Phi_2$ flux is conserved

if $A_1 > A_2$ then more field lines would have to "squeeze" thru a smaller area so if $\Phi_1 = \Phi_2$ then $\frac{d\Phi_1}{dt} = \frac{d\Phi_2}{dt} = \frac{d\Phi}{dt}$

now apply Faraday's law to each coil

$$\mathcal{E}_1 = -N_1 \frac{d\Phi}{dt} \quad \text{and} \quad \mathcal{E}_2 = -N_2 \frac{d\Phi}{dt}$$

$$\therefore \boxed{\frac{\mathcal{E}_1}{N_1} = \frac{\mathcal{E}_2}{N_2}} \quad \text{transformer equation}$$

ex: say we want $\mathcal{E}_1 = 10V$ and $\mathcal{E}_2 = 100V$

$$\frac{\mathcal{E}_1}{N_1} = \frac{\mathcal{E}_2}{N_2}$$

$$\frac{\mathcal{E}_1}{\mathcal{E}_2} = \frac{N_1}{N_2} = \frac{10}{100} = 0.1$$

if we build transforme with 100 turns in primary ($N_1 = 100$) then the secondary will need $N_2 = \frac{N_1}{0.1} = 10N_1 = 1000$ turns

this is called a "step up" transformer.

what about current?

Power has to be the same in both primary and secondary!

Power $= I \cdot V$ is energy/sec and energy is always conserved!

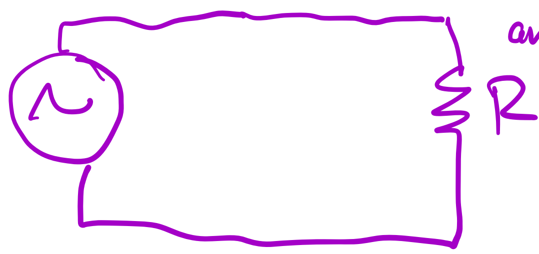
so $I_1 \mathcal{E}_1 = I_2 \mathcal{E}_2$ but $\mathcal{E}_2 = \mathcal{E}_1 N_2 / N_1$

so $I_1 \mathcal{E}_1 = I_2 \dfrac{\mathcal{E}_1 N_2}{N_1} \Rightarrow \boxed{N_1 I_1 = N_2 I_2}$

in previous example, $N_1 = 100$, $N_2 = 10$ so $I_1 > I_2$

step up transformer trades low V, high I for
high V, low I

How can we use this?



any power source pushes current thru load R
can we neglect resistance of wires?

loads usually have resistance $R \sim 100$'s Ω or more

copper wire: 20 gauge = 0.032 in diameter
= 0.81 mm "

resistance is 10.15 Ω per 1000 ft
so in lab, length of wire ~ 1-10 ft
~~total~~ resistance of wire is .01Ω for 1 ft
.1Ω for 10 ft

R_{wires} is negligible compared to load resistances

14 gauge wire: 0.064 in = 1.6 mm
resistance = 2.53 Ω per 1000 ft

now build power plant that generates current and sends it ~20 miles on 14 gauge copper wire

20 miles = 10^5 ft

resistance of 14 gauge wire = $\frac{2.53\ \Omega}{1000\ \text{ft}} \times 10^5\ \text{ft} = 253\ \Omega$

microwave oven has internal resistance $R_L = 24\ \Omega$

if $V = 120V$ then it draws 5A

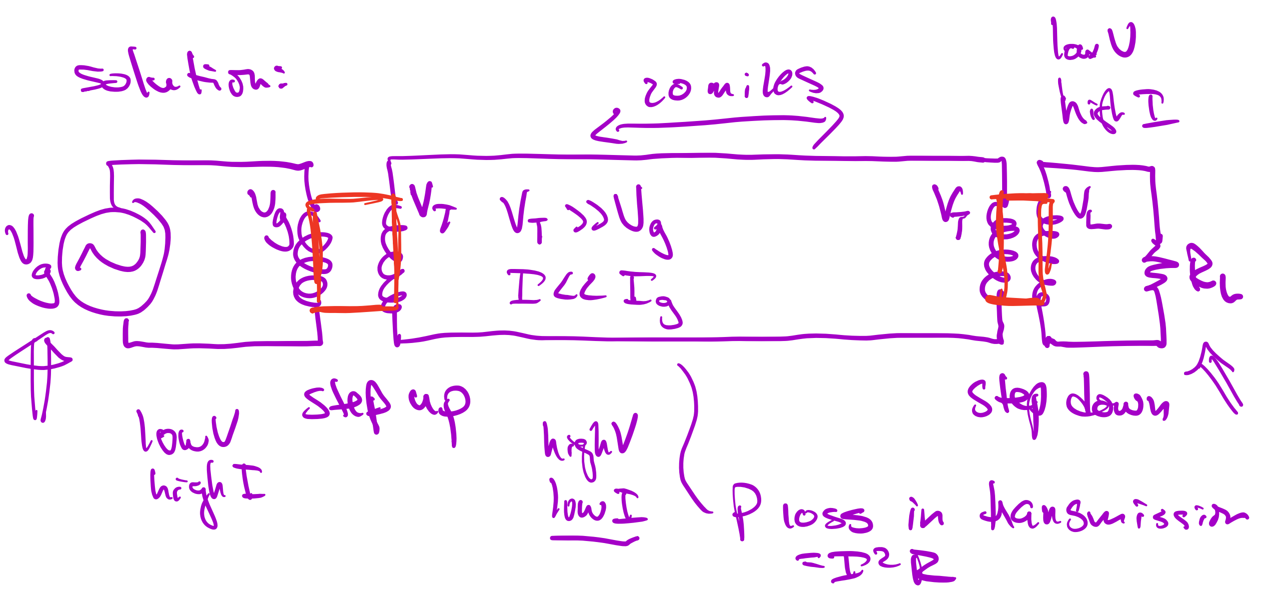
Power needed is $I \cdot V = 600W$

if wire resistance is $253\ \Omega$, then power dissipated in wires is

$$P_R = I^2R = 5^2 \cdot 253 \cong 6{,}300W$$

so most of the power generated goes into heating transmission wires!

solution:



basic idea for transformers: transmit at high voltage / low current to minimize current losses: $P = I^2 R$